

ON THE CARDINALITY OF HAUSDORFF SPACES AND H-CLOSED SPACES

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ABSTRACT. We introduce the cardinal invariant $aL'(X)$ and show that $|X| \leq 2^{aL'(X)\chi(X)}$ for any Hausdorff space X (a corollary of Theorem 4.4). This invariant has the properties a) $aL'(X) = \aleph_0$ if X is H-closed, and b) $aL(X) \leq aL'(X) \leq aL_c(X)$. Theorem 4.4 then gives a new improvement of the well-known Hausdorff bound $2^{L(X)\chi(X)}$ from which it follows that $|X| \leq 2^{\psi_c(X)}$ if X is H-closed (Dow/Porter [5]). The invariant $aL'(X)$ is constructed using convergent open ultrafilters and an operator $c : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ with the property $clA \subseteq c(A) \subseteq cl_\theta(A)$ for all $A \subseteq X$. As a comparison with this open ultrafilter approach, in §3 we additionally give a κ -filter variation of Hodel's proof [10] of the Dow-Porter result. Finally, for an infinite cardinal κ , in §5 we introduce κ wH-closed spaces, $\kappa H'$ -closed spaces, and $\kappa H''$ -closed spaces. The first two notions generalize the H-closed property. Key results in this connection are that a) if κ is an infinite cardinal and X a κ wH-closed space with a dense set of isolated points such that $\chi(X) \leq \kappa$, then $|X| \leq 2^\kappa$, and b) if X is $\kappa H'$ -closed or $\kappa H''$ -closed then $aL'(X) \leq \kappa$. This latter result relates these notions to the invariant $aL'(X)$ and the operator c .

1. INTRODUCTION.

A space X is *H-closed* if every open cover \mathcal{V} of X has a finite subfamily \mathcal{W} such that $X = \bigcup_{W \in \mathcal{W}} clW$. In 1982 Dow and Porter [5] used H-closed extensions of discrete spaces to demonstrate that $|X| \leq 2^{\chi(X)}$ (in fact, $|X| \leq 2^{\psi_c(X)}$) for any H-closed space X . The technique was simplified in Porter [15], and in 2006 Hodel [10] gave a proof of the Dow-Porter result using κ -nets.

A natural general question is the following:

Question 1.1. *Does there exist a strengthening of Arhangel'skiĭ's cardinal inequality $|X| \leq 2^{L(X)\chi(X)}$ [2] for a general Hausdorff space X for which it follows as a corollary that $|X| \leq 2^{\chi(X)}$ if X is H-closed?*

This question was asked by Angelo Bella in personal communication with the second author. Another way to ask this is, does there exist a property \mathcal{P} of a Hausdorff space X that a) generalizes both the H-closed property and the Lindelöf property simultaneously, and b) $|X| \leq 2^{\chi(X)}$ for spaces X with property \mathcal{P} ?

As both H-closed spaces and Lindelöf spaces are *almost-Lindelöf* (that is, every open cover has a countable subfamily whose closures cover), the property “almost-Lindelöf” would seem to be a suitable candidate. However, in 1998 Bella and

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Yaschenko [4] showed that if κ is a non-measurable cardinal then there exists an almost-Lindelöf, first-countable Hausdorff space X such that $|X| > \kappa$. Thus, $|X| \leq 2^{\chi(X)}$ does not hold for all almost-Lindelöf Hausdorff spaces X . (It does, however, hold if the space is additionally Urysohn [3]). In 1988, Bella and Cammaroto [3] gave the bound $2^{aL_c(X)t(X)\psi_c(X)}$ for the cardinality of a Hausdorff space X , where $aL_c(X)$ is defined before Definition 2.6 below. As $aL_c(X) \leq L(X)$, this suggests that “ $aL_c(X) = \aleph_0$ ” might be the required property \mathcal{P} . Yet the Katětov H-closed extension $\kappa\omega$ of the discrete space ω is an example of an H-closed space for which $aL_c(X) = \mathfrak{c} > \aleph_0$, demonstrating that the property $aL_c(X) = \aleph_0$ does not hold for all H-closed spaces.

In this study we construct a cardinal invariant $aL'(X)$ such that a) $|X| \leq 2^{aL'(X)\chi(X)}$ for a Hausdorff space X (Theorem 4.4 gives a slightly stronger version of this statement), b) $aL(X) \leq aL'(X) \leq aL_c(X)$ (Proposition 2.10), and c) $aL'(X) = \aleph_0$ if X is H-closed (follows from Corollary 3.5). Thus, the property “ $aL'(X) = \aleph_0$ ” is the required property \mathcal{P} above. Theorem 4.4 then gives a new bound on the cardinality of a Hausdorff space that is strong enough to capture the H-closed bound $2^{\chi(X)}$ given by Dow and Porter.

For an open set U in a space X , convergent open ultrafilters are used to define a set \widehat{U} (Definition 2.2) such that $U \subseteq \widehat{U} \subseteq clU$. Using the set \widehat{U} , we then define an operator $c : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ that satisfies $clA \subseteq c(A) \subseteq cl_\theta(A)$ for all $A \subseteq X$. Both \widehat{U} and the function c have relationships to the Iliadis absolute EX that are outlined in §2. After this set-up, the invariant $aL'(X)$ is defined as in Definition 2.6.

In Theorem 3.4, we give the following characterization of H-closed spaces, which is of interest in its own right: a space X is H-closed if and only if for every open cover \mathcal{V} of X there exists $\mathcal{W} \in [\mathcal{V}]^{<\omega}$ such that $X = \bigcup_{W \in \mathcal{W}} \widehat{W}$. Given that $\widehat{W} \subseteq clW$ for every open set W , this characterization is then a logically stronger property than the usual definition of the H-closed property. It follows naturally to define a cardinal invariant $L'(X)$ as the least infinite cardinal κ such that if \mathcal{V} is a cover of X then there exists $\mathcal{W} \in [\mathcal{V}]^{\leq \kappa}$ such that $X = \bigcup_{W \in \mathcal{W}} \widehat{W}$ (Definition 3.1). Theorem 3.4 shows that $L'(X) = \aleph_0$ if X is H-closed. We demonstrate that $L'(X)$ is hereditary on c -closed subsets (Proposition 3.2), from which it follows that $aL'(X) \leq L'(X) \leq L(X)$. Thus, given our main cardinality bound for general Hausdorff spaces (Theorem 4.4), we see that the following is a sufficient property of both Lindelöf and H-closed spaces X from which it follows that $|X| \leq 2^{\chi(X)}$: every open cover \mathcal{V} has a countable subfamily \mathcal{W} such that $X = \bigcup_{W \in \mathcal{W}} \widehat{W}$ (that is, $L'(X) = \aleph_0$). As $aL'(X) \leq L'(X)$, another such property (albeit weaker) is “ $aL'(X) = \aleph_0$ ”, as mentioned above.

In [10], Hodel gave a proof that $|X| \leq 2^{\psi_c(X)}$ for H-closed spaces X using the notion of a κ -net for a cardinal κ . This proof is different than previous proofs of this result given by Dow and Porter, and also different than the approach taken in Theorem 4.4 in this study. In §3 we use a filter characterization of H-closed spaces given in Theorem 2.21 and the c -adherence of a filter to give another proof that the cardinality of an H-closed space X is bounded by $2^{\psi_c(X)}$. This particular method

can be seen as a variation of the method used by Hodel [10] for nets. We present two examples at the end of §3.

In §4 we give the proof of our main result, Theorem 4.4, after establishing preliminary results in §2 – §4. The proof is fundamentally a standard closing-off argument. We use a theorem of Hodel (re-stated in Theorem 4.3) that gives a set-theoretic generalization of many such arguments. Typically the closure operator is used in a closing-off argument, or occasionally the θ -closure operator. We use the operator c referred to above.

In §5 we introduce two notions that generalize the H-closed property and a related third notion. The first is, for an infinite cardinal κ , the concept of a κ wH-closed space (Definition 5.5). This notion grows naturally out of recent work of Osipov in [14]. In Proposition 5.6 we give this characterization of H-closed: X is H-closed if and only if X is \aleph_0 wH-closed. A key result is Theorem 5.10, which states that if κ is an infinite cardinal and X is a κ wH-closed space with a dense set of isolated points such that $\chi(X) \leq \kappa$, then $|X| \leq 2^\kappa$. The second notion introduced in §5 is that of a $\kappa H'$ -closed space (Definition 5.12). Proposition 5.13 demonstrates that X is H-closed if and only if X is $\aleph_0 H'$ -closed. After defining $z(X) = \inf\{\kappa \geq \aleph_0 : X \text{ is } \kappa H' \text{-closed}\}$, it is shown in Corollary 5.16(a) that $aL'(X)^+ \leq z(X)$, thereby relating the notion of $\kappa H'$ -closed to concepts defined in previous sections. It follows immediately $|X| \leq 2^{z(X)\chi(X)}$ for any Hausdorff space X after applying Theorem 4.4. Finally, we introduce the property of $\kappa H''$ -closed in Definition 5.17 and, for a space X , we define the cardinal invariant $z'(X) = \inf\{\kappa \geq \aleph_0 : X \text{ is } \kappa H''\text{-closed}\}$. While it can be shown that $aL'(X) \leq z'(X)$ and thus $|X| \leq 2^{z'(X)\chi(X)}$ for any Hausdorff space X (Corollary 5.20), it is not guaranteed that a $\aleph_0 H''$ -closed space is H-closed. In fact, any countable space is $\aleph_0 H''$ -closed.

All spaces are assumed to be Hausdorff. For all undefined notions see Engelking [6], Juhász [12], or Porter-Woods [17]. Hodel's survey paper [10] also contains thorough discussion of many cardinal invariants and cardinality bounds related to those discussed in this study.

2. CONSTRUCTION OF THE CARDINAL FUNCTION $aL'(X)$.

Given a Hausdorff space X and an open set U of X , define

$$0U = \{\mathcal{U} : \mathcal{U} \text{ is a convergent open ultrafilter containing } U\}.$$

We recall the construction of the Iliadis absolute EX as the set of convergent open ultrafilters on X with the topology generated by the basis $\{0U : U \text{ is open in } X\}$. (See [17], Chapter 6, for example). Under this topology EX is an extremally disconnected, zero-dimensional, Tychonoff space. For each $\mathcal{U} \in EX$, let $k(\mathcal{U})$ be the unique convergent point of \mathcal{U} . We have the following basic facts concerning EX and the map $k : EX \rightarrow X$ (see [17] 6.6(e)(5), 6.8(d,f) and [16] 1.2(b)). Recall a subset A of a space X is an H -set if for every cover \mathcal{V} of A by sets open in X there exists $\mathcal{W} \in [\mathcal{V}]^{<\omega}$ such that $A \subseteq \bigcup_{W \in \mathcal{W}} cW$ and a space X is Katětov if X has a coarser H-closed topology.

Proposition 2.1. *For a open sets $U, V \subseteq X$ of a space X , and $\mathcal{U} \in EX$,*

- (a) *The map $k : EX \rightarrow X$ is a θ -continuous, perfect, irreducible, surjection,*
- (b) *$U \in \mathcal{U}$ iff $\text{int}(cl(U)) \in \mathcal{U}$ iff $\mathcal{U} \in 0U$, and thus $0U = 0(\text{int}(clU))$,*
- (c) *$k[0U] = cl(U)$,*
- (d) *$k^{\leftarrow}(k(\mathcal{U})) \subseteq 0U$ iff $k(\mathcal{U}) \in \text{int}(cl(U))$, and*
- (e) *If $B \subseteq EX$ is compact, then $k[B]$ is Katětov and an H -set.*
- (f) *$0(U \cap V) = 0U \cap 0V$ and $0(U \cup V) = 0U \cup 0V$.*

Let $b : X \rightarrow EX$ be an injective function such that $k \circ b = id_X$. That is, for all $x \in X$, $b(x)$ is an open ultrafilter converging to x . Denote the subspace $b[X]$ of EX by X_b . The space X_b is a *section* of EX [18] and is an extremally disconnected, Tychonoff space. We observe that $k|_{X_b} : X_b \rightarrow X$ is a bijection as X is Hausdorff.

Definition 2.2. For a space X , an open set U , and a section X_b of EX , define

$$\widehat{U}_b = \{x \in X : U \in b(x)\}.$$

We give several properties of \widehat{U}_b in Proposition 2.3. As is indicated in Proposition 2.3(a), \widehat{U}_b consists of a special set of closure points x of U having the stronger property that U is a member of the open ultrafilter $b(x)$. The set \widehat{U}_b will play a major role in the construction of the cardinal invariant $aL'(X)$ and in the proof of our main theorem, Theorem 4.4. In addition, for a space (X, τ) , $\{\widehat{U}_b : U \in \tau(X)\}$ forms a basis for a topology σ_b on X such that (X, σ_b) is homeomorphic to the section X_b (Proposition 2.5). Furthermore, Proposition 3.4 gives a characterization of H -closed spaces using sets of the form \widehat{U}_b . It is this characterization that will give the cardinality bound $2^{\psi_c(X)}$ for H -closed spaces as an immediate consequence of the general Hausdorff bound given in Theorem 4.4.

Proposition 2.3. *Let (X, τ) be a space, U, V open sets, and X_b be any section of EX . Then,*

- (a) *$U \subseteq \widehat{U}_b \subseteq clU$,*
- (b) *$\widehat{U}_b = k[0U \cap X_b]$,*
- (c) *$\widehat{(\overline{U \cap V})}_b = \widehat{U}_b \cap \widehat{V}_b$ and $\widehat{(\overline{U \cup V})}_b = \widehat{U}_b \cup \widehat{V}_b$,*
- (d) *$X \setminus \widehat{U}_b = \widehat{(X \setminus cl_X U)}_b$.*

Proof. (a) If $x \in U$, then U is a member of any open ultrafilter converging to x . Thus, $U \in b(x)$ and $x \in \widehat{U}_b$. This shows $U \subseteq \widehat{U}_b$. Suppose $x \in \widehat{U}_b$. Then $U \in b(x)$. If $x \in X \setminus clU$, then $X \setminus clU \in b(x)$ and thus $\emptyset = U \cap (X \setminus clU) \in b(x)$, a contradiction. Thus $x \in clU$ and $\widehat{U}_b \subseteq clU$.

(b) If $x \in \widehat{U}_b$, then $b(x) \in X_b$, $U \in b(x)$, and $b(x) \in 0U$. Since $k \circ b = id_X$, we see that $x = k(b(x))$, thus $x \in k[0U \cap X_b]$. This shows $\widehat{U}_b \subseteq k[0U \cap X_b]$. The reverse containment is similar.

(c) follows from (b) above and Proposition 2.1(f).

(d) If $x \in X \setminus \widehat{U}_b$, then $U \notin b(x)$ and therefore $X \setminus clU \in b(x)$. Thus $x \in \widehat{(X \setminus cl_X U)}_b$ and $X \setminus \widehat{U}_b \subseteq \widehat{(X \setminus cl_X U)}_b$. The reverse containment is identical. \square

Recall that the semiregularization $X(s)$ of a space X is the (Hausdorff) space with underlying set X and topology generated by the basis of regular-open sets in X .

Corollary 2.4. *Let X be a space, $U \in \tau(X)$, and $b : X \rightarrow EX$ be a section. Then $X_b = X(s)_b$.*

Proof. Note that $b : X(s) \rightarrow EX(s)$ is also a section and $EX = E(X(s))$. By 2.3(b), $\widehat{U}_b = k[0U \cap X_b] = k[0(int_X cl_X U \cap X_b)] = (int_X \widehat{cl_X U})_b$. Thus, $X_b = X(s)_b$. \square

Proposition 2.5. *Let (X, τ) be a space and X_b a section of EX . Then $\{\widehat{U}_b : U \in \tau(X)\}$ is a clopen base for an extremally disconnected, Tychonoff topology σ_b on X such that (X, σ_b) is homeomorphic to X_b .*

Proof. The proof follows from 2.3(b) and the fact that $k|_{X_b} : X_b \rightarrow X$ is a bijection. \square

For a space X and a section X_b of EX , define an operator $c_b : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ by

$$c_b(A) = \{x \in X : \widehat{U}_b \cap A \neq \emptyset \text{ for all } U \text{ such that } x \in U \in \tau(X)\}.$$

We say that $A \subseteq X$ is c_b -closed if $c_b(A) = A$.

Recall that for $A \subseteq X$, we define $aL(A, X)$ as the least infinite cardinal κ such that if \mathcal{V} is a cover of A by sets open in X then there exists $\mathcal{W} \in [\mathcal{V}]^{\leq \kappa}$ such that $A \subseteq \bigcup_{W \in \mathcal{W}} cl W$. The *almost Lindelöf degree* of X , denoted by $aL(X)$, is $aL(X, X)$. The *almost Lindelöf degree of X with respect to closed sets* is

$$aL_c(X) = \sup\{aL(C, X) : C \text{ is closed}\} + \aleph_0.$$

It is straightforward to see that $aL(X) \leq aL_c(X) \leq L(X)$ and that all three are identical if X is regular.

Definition 2.6. For a section X_b of EX , we define the cardinal invariant $aL^b(X)$, by $aL^b(X) = \sup\{aL(C, X) : C \text{ is } c_b\text{-closed}\} + \aleph_0$. As $aL^b(X)$ depends on the choice of section X_b , we define the unique cardinal invariant $aL'(X)$ by

$$aL'(X) = \min\{aL^b(X) : X_b \text{ is a section of } EX\}.$$

Let $X_{b'}$ be any section witnessing that $aL'(X) = aL^{b'}(X)$ and define the operator $c : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ by $c = c_{b'}$. We then refer to $aL'(X)$ as the *almost Lindelöf degree of X with respect to c -closed sets*.

Notation. For an open set U of X , we let \widehat{U} denote $\widehat{U}_{b'}$. For $A \subseteq X$, we let $A' = b'[A] \subseteq EX$. For a point x in a space X , let \mathcal{U}_x represent the open ultrafilter $b'(x)$. In general, throughout this study we will reserve the symbol “ \mathcal{U} ” to represent a convergent open ultrafilter.

The function c defined above is the main operator in the closing-off argument used to prove our main result, Theorem 4.4. We give properties of c below. For a subset $A \subseteq X$, we also give a characterization of $c(A)$ using EX in Theorem 2.15.

Proposition 2.7. *Let X be a space, and $A, B \subseteq X$.*

- (a) $A \subseteq c(A)$.
- (b) *if $A \subseteq B$ then $c(A) \subseteq c(B)$.*
- (c) $clA \subseteq c(A) \subseteq cl_\theta(A)$.
- (d) *if U is open, then $clU = c(U) \subseteq c(\widehat{U})$.*
- (e) *if X is regular then $clA = c(A) = cl_\theta(A)$.*
- (f) *If A is c -closed then A is closed.*

Proof. (a) If $x \in A$ and U is an open set containing x , then $x \in \widehat{U} \cap A$ by Proposition 2.3(a).

(b) If $x \in c(A)$ then $\emptyset \neq \widehat{U} \cap A \subseteq \widehat{U} \cap B$ for all open sets U containing x . This shows $x \in c(B)$.

(c) If $x \in clA$, then by Proposition 2.3(a), $\emptyset \neq U \cap A \subseteq \widehat{U} \cap A$ for all open sets U containing x . Thus $x \in c(A)$. And if $x \in c(A)$, then $\emptyset \neq \widehat{U} \cap A \subseteq clU \cap A$, also by Proposition 2.3(a). Thus, $x \in cl_\theta(A)$.

(d) As $clU = cl_\theta U$ for an open set U , the equality follows from (b). The containment follows from (a) and Proposition 2.3(a).

(e) As $clA = cl_\theta A$ for regular spaces, the result follows from (b).

(f) If A is c -closed and $x \in X \setminus A$, then there exists an open set U containing x such that $\emptyset = \widehat{U} \cap A \supseteq U \cap A$. Thus A is closed. \square

The following example provides a space X and a subset A such that $c(A) \neq cl_X(A)$.

Example 2.8. Consider the Katětov H-closed extension $\kappa\omega$ of ω with the discrete topology (cf. Ch 7 in [17]). Recall that $\kappa\omega(s) = \beta\omega$; that is, $\beta\omega$ is the underlying set of $\kappa\omega$. Also note that $\kappa\omega \setminus \omega$ is discrete and closed. By 9.11 in [7], the closed set $\beta\omega \setminus \omega$ contains a copy of $\beta\omega$. That is, there a countable discrete subspace A of $\beta\omega \setminus \omega$ such that $\beta A = \beta\omega$, in particular, $cl_{\beta\omega} A = \beta\omega$. Then A is a closed subset of $\kappa\omega$. Let $k : \beta\omega \rightarrow \kappa\omega$ denote the identity function. $\beta\omega$ is an extremally disconnected, Tychonoff space and the bijection k is perfect, irreducible, and θ -continuous. By 6.7(a) in [17], $EX = \beta\omega$ is the absolute of $X = \kappa\omega$ with $k : EX \rightarrow X$ the absolute map. There is only one injective function $b : X \rightarrow EX$ such that $k \circ b = id_X$. It follows that $c(A) = cl_{\beta\omega}(A)$. By 9.3 in [7], $c(A)$ has cardinality 2^c ; thus, $c(A) \neq cl_X(A) = A$. This example also illustrates that there can be a marked size difference between $c(A)$ and $cl(A)$. \square

Observe that for the space $X = \kappa\omega$ in Example 2.8, we have $\sigma_b \subsetneq \tau$, where τ is the topology on X . However, by 5.1(d) in [18], for a regular space X , we have that $\tau \subseteq \sigma_b$. Thus there is no universal containment relationship between τ and σ_b .

Let X be a space, $U \in \tau(X)$, and $b : X \rightarrow EX$ be a section. By 2.3(d), it follows that \widehat{U}_b is also closed in σ_b . The next example shows that \widehat{U}_b may not be c -closed.

Example 2.9. Let \mathcal{U} and \mathcal{V} be distinct free open ultrafilters on ω , i.e., distinct points in $\beta\omega \setminus \omega$. Let $\alpha\omega$ denote the compactification of ω (discrete topology) where \mathcal{U} and \mathcal{V} in $\beta\omega$ are identified as the point y . Let $X = \alpha\omega$. Then $EX = \beta\omega$ and

$k : EX \rightarrow X$ is the identity function on $EX \setminus \{\mathcal{U}, \mathcal{V}\}$ and $k(\mathcal{U}) = k(\mathcal{V}) = y$. Consider the section defined by the function

$$b : X \rightarrow EX : x \mapsto \begin{cases} x & x \neq y \\ \mathcal{U} & x = y. \end{cases}$$

For $A \in [\omega]^\omega$, let

$$o_\alpha A = A \cup \{\mathcal{W} \in \beta\omega \setminus \{\mathcal{U}, \mathcal{V}\} : A \in \mathcal{W}\} \cup \begin{cases} \emptyset & A \notin \mathcal{U} \cap \mathcal{V} \\ \{y\} & A \in \mathcal{U} \cap \mathcal{V} \end{cases}.$$

Then $\{o_\alpha A : A \in [\omega]^\omega\} \cup \{\{n\} : n \in \omega\}$ is a base for $\tau(\alpha\omega)$. In particular, a neighborhood base for y is $\{o_\alpha W : W \in \mathcal{U} \cap \mathcal{V}\}$. Also, note that for $A \in [\omega]^\omega$,

$$\widehat{o_\alpha A} = o_\alpha A \cup \begin{cases} \emptyset & A \notin \mathcal{U} \\ \{y\} & A \in \mathcal{U} \end{cases}.$$

Let $T \in \mathcal{V} \setminus \mathcal{U}$. Then $y \notin \widehat{o_\alpha T}$. For $W \in \mathcal{U} \cap \mathcal{V}$, $y \in o_\alpha W$ and $\widehat{o_\alpha W} \cap \widehat{o_\alpha T} \supseteq W \cap T \neq \emptyset$. That is, $c(\widehat{o_\alpha T}) = \widehat{o_\alpha T} \cup \{y\} \neq \widehat{o_\alpha T}$. \square

In view of Proposition 2.7(f) and the fact that any space is c -closed in itself, we have the following:

Corollary 2.10. *For any space X , $aL(X) \leq aL'(X) \leq aL_c(X) \leq L(X)$.*

For a space X , we define a cardinal invariant $t_c(X)$ related to the tightness $t(X)$. While $t_c(X)$ and $t(X)$ appear to be incomparable, Proposition 2.12 shows that $t_c(X)$ is bounded above by the character $\chi(X)$.

Definition 2.11. For a space X , the c -tightness of X , $t_c(X)$, is defined as the least cardinal κ such that if $x \in c(A)$ for some $x \in X$ and $A \subseteq X$, then there exists $B \in [A]^{\leq \kappa}$ such that $x \in c(B)$.

Note that $t(\kappa\omega) = \aleph_0$ and $t_c(\kappa\omega) = t(\beta\omega) = \mathfrak{c}$. This shows that $t(\kappa\omega)$ and $t_c(\kappa\omega)$ are not equal.

Proposition 2.12. *For any space X , $t_c(X) \leq \chi(X)$. If X is regular then $t_c(X) = t(X)$.*

Proof. To show $t_c(X) \leq \chi(X)$, let $\kappa = \chi(X)$ and let $x \in c(A)$. Let \mathcal{N} be an open neighborhood base at x such that $|\mathcal{N}| = \kappa$. For all $N \in \mathcal{N}$ there exists $a_N \in \widehat{N} \cap A$. Let $D = \{a_N : N \in \mathcal{N}\} \in [A]^{\leq \kappa}$. We show $x \in c(D)$. Let U be an open set containing x . There exists $N \in \mathcal{N}$ such that $x \in N \subseteq U$. As $a_N \in \widehat{N} \cap A$, we have $N \in \mathcal{U}_{a_N}$. As \mathcal{U}_{a_N} is an open filter, we have that $U \in \mathcal{U}_{a_N}$ and $a_N \in \widehat{U} \cap D$. This shows $x \in c(D)$ and $t_c(X) \leq \kappa$. If X is regular then $t_c(X) = t(X)$ follows from Proposition 2.7(e). \square

In Theorem 2.15, we give a characterization of $c(A)$ for a subset $A \subseteq X$ in terms of the absolute EX . This is one of several results below that describe how $c(A)$ relates to the broader framework of EX .

Let $\mathcal{K} = \{k^{\leftarrow}(p) : p \in X\}$, where $k : EX \rightarrow X$ is as in 2.1(a). For $A \subseteq EX$, define $cl_{\mathcal{K}}A = A \cup \bigcup \{K \in \mathcal{K} : \text{if } K \subseteq 0U \text{ for } U \in \tau(X), 0U \cap A \neq \emptyset\}$.

Lemma 2.13. *For $A \subseteq EX$, $cl_X k[A] \subseteq k[cl_{EX} A] \subseteq k[cl_{\mathcal{K}} A] \subseteq cl_{\theta} k[A]$.*

Proof. As k is a closed function, we immediately have that $cl_X k[A] \subseteq k[cl_{EX} A]$. To show $k[cl_{EX} A] \subseteq k[cl_{\mathcal{K}} A]$, it suffices to show that $cl_{EX} A \subseteq cl_{\mathcal{K}} A$. Let $\mathcal{U} \in cl_{EX} A$ and $K = k^{\leftarrow}(k(\mathcal{U})) \subseteq 0U$ for some $U \in \tau(X)$. Then $\mathcal{U} \in 0U$ and $0U \cap A \neq \emptyset$ as $\mathcal{U} \in cl_{EX} A$. Thus, $\mathcal{U} \in K \subseteq cl_{\mathcal{K}} A$.

Now, let $p \in k[cl_{\mathcal{K}} A]$ and $p \in U \in \tau(X)$. Then, by Proposition 2.1(d), $k^{\leftarrow}(p) \subseteq 0U$. So, $0U \cap A \neq \emptyset$ and $\emptyset \neq k[0U \cap A] \subseteq k[0U] \cap k[A] \subseteq cl_X(U) \cap k[A]$. Therefore, $x \in cl_{\theta} k[A]$. \square

Lemma 2.14. *For $A \subseteq EX$,*

$$cl_{\mathcal{K}} A = \bigcup \{K \in \mathcal{K} : K \cap cl_{\theta} A \neq \emptyset\} = \bigcup k^{\leftarrow}[k[cl_{\theta} A]] = k^{\leftarrow}[k[cl_{EX} A]].$$

Proof. Suppose $K \cap cl_{\theta} A = \emptyset$ for $K \in \mathcal{K}$. Then for each $\mathcal{U} \in K$, there is $U_{\mathcal{U}}$ such that $0(U_{\mathcal{U}}) \cap A = \emptyset$ and $\{0(U_{\mathcal{U}}) : \mathcal{U} \in K\}$ is an open cover of the compact set K . There exists $\mathcal{U}_1, \dots, \mathcal{U}_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^n 0(U_{\mathcal{U}_i}) = 0\left(\bigcup_{i=1}^n U_{\mathcal{U}_i}\right),$$

by 2.1(f). Let $U = \bigcup_{i=1}^n U_{\mathcal{U}_i}$. As $K \subseteq 0(U)$ and $0(U) \cap A = \emptyset$, we see that $K \cap cl_{\mathcal{K}} A = \emptyset$. Conversely, suppose $\mathcal{U} \in K \cap cl_{\theta} A$ and $K \subseteq 0(U)$. Then $\mathcal{U} \in 0(U)$ and $0(U) \cap A \neq \emptyset$. Thus $K \subseteq cl_{\mathcal{K}} A$. This shows the first equality.

To show $\bigcup \{K \in \mathcal{K} : K \cap cl_{\theta} A \neq \emptyset\} = \bigcup k^{\leftarrow}[k[cl_{\theta} A]]$, it suffices to note that if $\mathcal{U} \in cl_{\theta} A$, then $k^{\leftarrow}[k(\mathcal{U})] \cap cl_{\theta} A \neq \emptyset$ and $k^{\leftarrow}[k(\mathcal{U})] \in \mathcal{K}$. The equality $\bigcup k^{\leftarrow}[k[cl_{\theta} A]] = k^{\leftarrow}[k[cl_{EX} A]]$ follows as EX is Tychonoff. \square

Theorem 2.15. *Let X be a space and $A \subseteq X$. Then $c(A) = k[cl_{\mathcal{K}} A'] = k[cl_{EX} A']$.*

Proof. We first show the first equality. Clearly, $A \subseteq c(A) \cap k[cl_{\mathcal{K}} A']$. Let $p \in c(A) \setminus A$ and $k^{\leftarrow}(p) \subseteq 0U$ where $U \in \tau(X)$. By 2.1(d), $p \in \text{int}(cl(U))$. There is $a \in A$ such that $\text{int}(cl(U)) \in \mathcal{U}_a$. By 2.1(b), $U \in \mathcal{U}_a \subseteq k^{\leftarrow}(a)$. Thus, $\mathcal{U}_a \in 0(\text{int}(cl(U))) = 0(U)$. That is, $\mathcal{U}_a \in 0(U) \cap A'$. This shows that $k^{\leftarrow}(p) \subseteq cl_{\mathcal{K}} A'$ and $p \in k[cl_{\mathcal{K}} A']$. Conversely suppose $p \in k[cl_{\mathcal{K}} A'] \setminus A$. Let $p \in U \in \tau(X)$. Then $k^{\leftarrow}(p) \subseteq 0(U)$ and $0(U) \cap A' \neq \emptyset$. There is $a \in A$ such that $\mathcal{U}_a \in 0(U)$. Thus, $U \in \mathcal{U}_a$ and $p \in c(A)$. This shows that $c(A) = k[cl_{\mathcal{K}} A']$.

To show the second equality, note that by Lemma 2.14, we have

$$c(A) = k[cl_{\mathcal{K}} A'] = k[k^{\leftarrow}[k[cl_{EX} A']]] = k[cl_{EX} A'].$$

\square

As the map $k : EX \rightarrow X$ is always a closed map, we have the following corollary to Theorem 2.15.

Corollary 2.16. *For any space X and every $A \subseteq X$, $c(A)$ is closed subset of X .*

By Theorem 2.15 and Proposition 2.7(c), we also have the following corollary. We see that Corollary 2.17 is stronger than Proposition 2.7(c) and demonstrates how $c(A)$ sits between $cl_X A$ and $cl_{\theta} A$ in terms of the absolute EX .

Corollary 2.17. *Let X be a space and $A \subseteq X$. Then $cl_X A \subseteq k[cl_{EX} A'] = c(A) \subseteq cl_\theta A$.*

Our next corollary to Theorem 2.15 demonstrates that the c -closure of a subset of an H-closed space is both Katětov and an H-set. This result should be compared with Lemma 3.10 which gives different conditions under which a subset of an H-closed space is an H-set and the result from [11] that the θ -closure of a subset of an H-closed space is an H-set.

Corollary 2.18. *If X is H-closed and $A \subseteq X$, then $c(A)$ is Katětov and an H-set.*

Proof. As $c(A) = k[cl_{EX} A']$ by Theorem 2.15 and since the absolute EX is compact when X is H-closed ([17] 6.9(b)(1)), we have that $cl_{EX} A'$ is compact and $c(A)$ is Katětov and an H-set by 2.1(e). \square

Another consequence of Theorem 2.15 are the following properties of the c -closure operator and a new characterization of H-closed spaces.

Proposition 2.19. *Let X be a space and A, B subsets of X .*

- (a) *If $A \subseteq B$, then $c(A) \subseteq c(B)$.*
- (b) *$c(A \cap B) \subseteq c(A) \cap c(B)$.*
- (c) *$c(A \cup B) = c(A) \cup c(B)$.*

Proof. (a) is immediate from 2.15, and (b) follows from (a). For (c), note that $c(A \cup B) = k[cl_{EX}(A \cup B)'] = k[cl_{EX}(A' \cup B')'] = k[cl_{EX}(A') \cup cl_{EX}(B')] = k[cl_{EX}(A')] \cup k[cl_{EX}(B')] = c(A) \cup c(B)$. \square

Definition 2.20. Let \mathcal{F} be a filter base on a space X . We define the c -adherence of \mathcal{F} , denoted as $a_c(\mathcal{F})$, as $\cap\{c(F) : F \in \mathcal{F}\}$. By 2.7(c), it follows that $a(\mathcal{F}) \subseteq a_c(\mathcal{F}) \subseteq a_\theta(\mathcal{F})$.

We will use the concept of c -adherence to obtain a new characterization of H-closed spaces in the next result.

Theorem 2.21. *Let X be a space. Then X is H-closed iff for every filter base \mathcal{F} on X , $a_c(\mathcal{F}) \neq \emptyset$.*

Proof. Let \mathcal{F} be a filter base on X . To show X is H-closed, it suffices to show that $a_\theta(\mathcal{F}) \neq \emptyset$. But $c(F) \subseteq cl_\theta(F)$ for each $F \in \mathcal{F}$ and $a_c(\mathcal{F}) \neq \emptyset$. Thus, $a_\theta(\mathcal{F}) \neq \emptyset$. Conversely suppose X is H-closed. Let \mathcal{F} be a filter base on X . Then $\{cl_{EX} F' : F \in \mathcal{F}\}$ is a filter base of compact subsets on EX . Thus, there is $p \in \cap\{cl_{EX} F' : F \in \mathcal{F}\}$. It follows that $k(p) \in a_c(\mathcal{F})$. \square

In the next section we will develop further connections between the H-closed property, the operator c , and the set \hat{U} for an open set $U \subseteq X$.

3. H-CLOSED SPACES.

For a space X we define $L'(X)$, a cardinal invariant related to $aL'(X)$, and show in Proposition 3.2 that it is hereditary on c -closed subsets. The filter characterization of H-closed spaces used in Theorem 2.21 using c -adherence of a filter in

conjunction with a variation of a method used by Hodel [10] for nets provides a direct path for proving that the cardinality of an H-closed space X is bounded by $2^{\psi_c(X)}$.

Definition 3.1. For a subset $A \subseteq X$, define $L'(A, X)$ as the least cardinal κ such that for every cover \mathcal{V} of A by sets open in X there exists $\mathcal{W} \in [\mathcal{V}]^{\leq \kappa}$ such that $A \subseteq \bigcup_{W \in \mathcal{W}} \widehat{W}$. Set $L'(X) = L'(X, X)$.

Proposition 3.2. Let X be a space. If $A \subseteq X$ is c -closed, then $L'(A, X) \leq L'(X)$.

Proof. Let $\kappa = L'(X)$ and let \mathcal{V} be a cover of A by sets open in X . As A is c -closed, for all $x \in X \setminus A$, there exists an open set W_x containing x such that $a \notin \widehat{W}_x$ for all $a \in A$. Let $\mathcal{W} = \{W_x : x \in X \setminus A\}$. Then $\mathcal{W} \cup \mathcal{V}$ is an open cover of X . As $L'(X) = \kappa$, there exists $\mathcal{W}' \in [\mathcal{W}]^{\leq \kappa}$ and $\mathcal{V}' \in [\mathcal{V}]^{\leq \kappa}$ such that

$$X = \bigcup_{W \in \mathcal{W}'} \widehat{W} \cup \bigcup_{V \in \mathcal{V}'} \widehat{V}.$$

Suppose there exists $a \in A \cap \bigcup_{W \in \mathcal{W}'} \widehat{W}$. Then there exists $W \in \mathcal{W}'$ such that $a \in \widehat{W}$, a contradiction. Thus $A \cap \bigcup_{W \in \mathcal{W}'} \widehat{W} = \emptyset$ and $A \subseteq \bigcup_{V \in \mathcal{V}'} \widehat{V}$. This shows $L'(A, X) \leq \kappa$. \square

Corollary 3.3. For any space X , $aL'(X) \leq L'(X) \leq L(X)$.

Proof. To show the first inequality, let $\kappa = L'(X)$, let A be a c -closed subset of X , and let \mathcal{V} be a cover of A by sets open in X . As $L'(A, X) \leq \kappa$ by Proposition 3.2, there exists $\mathcal{W} \in [\mathcal{V}]^{\leq \kappa}$ such that $A \subseteq \bigcup_{W \in \mathcal{W}} \widehat{W}$. By Proposition 2.3(a), we see that $A \subseteq \bigcup_{W \in \mathcal{W}} \widehat{W} \subseteq \bigcup_{W \in \mathcal{W}} clW$. This shows $aL'(X) \leq \kappa$. To see that $L'(X) \leq L(X)$, just observe again by Proposition 2.3a that $V \subseteq \widehat{V}$ for every member V of an open cover of X . \square

In addition to Theorem 2.21, we obtain another new characterization of H-closed spaces.

Theorem 3.4. A space X is H-closed if and only if for every open cover \mathcal{V} of X there exists a finite family $\mathcal{W} \subseteq \mathcal{V}$ such that $X = \bigcup_{W \in \mathcal{W}} \widehat{W}$.

Proof. Let \mathcal{V} be an open cover of X and suppose there exists a finite family $\mathcal{W} \subseteq \mathcal{V}$ such that $X = \bigcup_{W \in \mathcal{W}} \widehat{W}$. Then, by Proposition 2.3(a), $X = \bigcup_{W \in \mathcal{W}} clW$, showing X H-closed.

Suppose now that X is H-closed and let \mathcal{V} be an open cover of X . As X is H-closed, there is a finite family $\mathcal{W} \in \mathcal{V}$ such that $X = \bigcup_{W \in \mathcal{W}} clW$. Suppose by way of contradiction that there exists $x \in X \setminus (\bigcup_{W \in \mathcal{W}} \widehat{W})$. Then, by Proposition 2.3(d),

$$x \in X \setminus \left(\bigcup_{W \in \mathcal{W}} \widehat{W} \right) = \bigcap_{W \in \mathcal{W}} (X \setminus \widehat{W}) = \bigcap_{W \in \mathcal{W}} (X \setminus clW).$$

Then, $X \setminus clW$ is a member of the open ultrafilter \mathcal{U}_x for all $W \in \mathcal{W}$. It follows by the finite intersection property that

$$\emptyset = \bigcap_{W \in \mathcal{W}} (X \setminus clW) \in \mathcal{U}_x.$$

As this is a contradiction, we see $X = \bigcup_{W \in \mathcal{W}} \widehat{W}$. \square

We have the following immediate corollary of Theorem 3.4.

Corollary 3.5. *If X is H-closed then $L'(X) = \aleph_0$.*

For the space $X = \kappa\omega$ in Example 2.8, we note by 3.5 that $L'(X) = \aleph_0$. Yet $L(X) = 2^c$. Furthermore, since $X_b = X(s)_b = \beta\omega$ for the section X_b in that example, it follows that $L(X(s)) = \aleph_0$.

We now present an example of an H-closed space X and a subset A such that $cl_X(A) \neq c(A) \neq cl_\theta(A)$ showing that 2.7(c) is the best general result.

Example 3.6. We use Urysohn's space \mathbb{U} defined in 1925 to show that the converse of Proposition 2.7(c) is not true in the setting of H-closed spaces. Let \mathbb{Z} denote the set of all integers with the discrete topology and \mathbb{N} denote the subspace of positive integers. For the set $\mathbb{U} = \mathbb{N} \times \mathbb{Z} \cup \{\pm\infty\}$, a subset $U \subseteq \mathbb{U}$ is defined to be open if $+\infty \in U$ (resp. $-\infty \in U$) implies for some $k \in \mathbb{N}$, $\{(n, m) : n \geq k, m \in \mathbb{N}\} \subseteq U$ (resp. $\{(n, -m) : n \geq k, m \in \mathbb{N}\} \subseteq U$) and if $(n, 0) \in U$ implies for some $k \in \mathbb{N}$, $\{(n, \pm m) : m \geq k\} \subseteq U$. The space \mathbb{U} is first countable, minimal Hausdorff (H-closed and semiregular) but is not compact as $A = \{(n, 0) : n \in \mathbb{N}\}$ is an infinite, closed discrete subset. Let $k : E\mathbb{U} \rightarrow \mathbb{U}$ be the absolute map from the absolute $E\mathbb{U}$ to \mathbb{U} . Let $\mathcal{U} \in k^{\leftarrow}(\infty)$ such that $\mathbb{N} \times \{2\} \in \mathcal{U}$; thus, $\mathcal{U} \rightarrow \infty$. Let $\mathcal{V} \in k^{\leftarrow}(-\infty)$ such that $\mathbb{N} \times \{-2\} \in \mathcal{V}$; thus, $\mathcal{V} \rightarrow -\infty$. For $n \in \mathbb{N}$, let $\mathcal{U}_n \in k^{\leftarrow}((n, 0))$ such that $\{n\} \times \mathbb{N} \in \mathcal{U}_n$; thus, $\mathcal{U}_n \rightarrow (n, 0)$. Define $b : \mathbb{U} \rightarrow E\mathbb{U}$ by $b(\infty) = \mathcal{U}$, $b(-\infty) = \mathcal{V}$, $b((n, 0)) = \mathcal{U}_n$, and for $(n, m) \in \mathbb{N} \times \mathbb{Z} \setminus \mathbb{N} \times \{0\}$, $b(n, m) = \{U \in \tau(\mathbb{U}) : (n, m) \in U\}$. It follows that $cl_{\mathbb{U}}(A) = A$ and $cl_\theta(A) = A \cup \{\pm\infty\}$. By 2.7(c), it follows $A \subseteq c(A) \subseteq A \cup \{\pm\infty\}$. To show that $\infty \in c(A)$, for $n \in \mathbb{N}$, let $T_n = (\mathbb{N} \setminus \{1, 2, \dots, n\}) \times \mathbb{N}$. A basic open set containing ∞ is $T_n \cup \{\infty\}$. As $\{n+1\} \times \mathbb{N} \in \mathcal{U}_{n+1} = b(n+1, 0)$, $T_n \in \mathcal{U}_{n+1}$ and $\hat{T}_n \cap A \neq \emptyset$. A similar argument shows that $-\infty \notin c(A)$. Thus, $c(A) = A \cup \{\infty\}$ and this shows that $cl_X(A) \neq c(A) \neq cl_\theta(A)$. Also, note that both $c(A)$ and $cl_\theta(A)$ are H-sets. \square

Definition 3.7. Let X be a space, κ an infinite cardinal, and $A \subseteq X$. A is κ -H-closed if for each open (in X) cover \mathcal{C} of A such that $|\mathcal{C}| \leq \kappa$, there is a finite subfamily $\mathcal{D} \subseteq \mathcal{C}$ such that $A \subseteq \bigcup_{\mathcal{D}} cl_X(U \cap A)$.

We note that in particular, a Hausdorff space X is ω -H-closed iff X is feebly compact.

We prove the following lemmas. The key lemma is Lemma 3.10, which is of interest on its own.

Lemma 3.8. *Let X be a space, κ an infinite cardinal, and $A \subseteq X$. If for each filter base $\mathcal{F} \in [[A]^{\leq \kappa}]^{\leq \kappa}$, $a_c(\mathcal{F}) \cap A \neq \emptyset$, then A is a κ -H-closed.*

Proof. Let \mathcal{C} be an open cover of A by sets open in X and suppose that \mathcal{C} is closed under finite unions and suppose $|\mathcal{C}| \leq \kappa$. For each $V \in \mathcal{C}$, assume there is $p_V \in A \setminus cl(V \cap A)$. Let $B_V = \{p_U : V \subseteq U \in \mathcal{C}\}$. For $T, S \in \mathcal{C}$, $B_T \cap B_S \supseteq B_{T \cup S}$. Then $\mathcal{F} = \{B_V : V \in \mathcal{C}\}$ is a filter base on A . Thus, there is a point $p \in a_c(\mathcal{F}) \cap A$. There is $T \in \mathcal{C}$ such that $p \in T$. Now, $B_T \subseteq A \setminus cl(T \cap A) \subseteq X \setminus cl(T \cap A)$. By Propositions 2.7(d) and 2.19(a), using that $X \setminus cl(T \cap A)$ is open, we have that $p \in c(B_T) \subseteq c(X \setminus cl(T)) = cl(X \setminus cl(T)) \subseteq X \setminus T$, a contradiction as $p \in T$. \square

The small filter base method presented in the above lemma stands in contrast to the open ultrafilter techniques frequently used in H-closed settings. An immediate consequence is this corollary.

Corollary 3.9. *Let X be a space and $A \subseteq X$. If for every filter base \mathcal{F} on A , $a_c(\mathcal{F}) \cap A \neq \emptyset$, then the subspace A is H-closed.*

Lemma 3.10. *Let X be an H-closed space, κ an infinite cardinal, $A \subseteq X$, and $\psi_c(X) \leq \kappa$. If for every filter base \mathcal{F} on $[[A^{\leq \kappa}]^{\leq \kappa}]$, $a_c(\mathcal{F}) \cap A \neq \emptyset$. Then A is an H-set.*

Proof. Let \mathcal{G} be an open filter that meets A . We can assume that \mathcal{G} is maximal with respect to meeting A . As X is H-closed, there is $p \in a(\mathcal{G})$. The goal is to show that $p \in A$. Assume that $p \notin A$. There is a family $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ of open neighborhoods of p such that $\bigcap_\kappa cl(V_\alpha) = \{p\}$. For each V_α , as $p \notin cl(X \setminus cl(V_\alpha))$, $X \setminus cl(V_\alpha) \notin \mathcal{G}$. There is some $G_\alpha \in \mathcal{G}$ such that $(X \setminus cl(V_\alpha)) \cap G_\alpha \cap A = \emptyset$. Thus, $G_\alpha \cap A \subseteq cl(V_\alpha)$ and $cl(G_\alpha \cap A) \subseteq cl(V_\alpha)$. Assume, by way of contradiction, that $A \cap \bigcap cl(G_\alpha \cap A) = \emptyset$. Thus, $\{X \setminus cl(G_\alpha \cap A) : \alpha < \kappa\}$ is open cover of A . As A is κ -H-closed by Lemma 3.8, there is a finite set $F \in [\kappa]^{<\omega}$ such that $A \subseteq \bigcup_F cl((X \setminus cl(G_\alpha \cap A)) \cap A)$. There is a $G \in \mathcal{G}$ such that $G \subseteq \bigcap_F G_\alpha$. For $\alpha \in F$, $G \cap A \subseteq G_\alpha \cap A$ implying that $X \setminus cl(G_\alpha \cap A) \subseteq X \setminus cl(G \cap A)$ and $(X \setminus cl(G_\alpha \cap A)) \cap A \subseteq (X \setminus cl(G \cap A)) \cap A$. Thus $A \subseteq cl((X \setminus cl(G \cap A)) \cap A) = cl(A \setminus cl_A(G \cap A))$ implying that $A \subseteq cl_A(A \setminus cl_A(G \cap A)) = A \setminus int_A cl_A(G \cap A)$, a contradiction as $int_A cl_A(G \cap A) \cap A \neq \emptyset$. \square

Theorem 3.11. *Let X be H-closed, κ an infinite cardinal, and $\psi_c(X) \leq \kappa$. Then $|X| \leq 2^\kappa$.*

Proof. For each $x \in X$, let $\{V(\alpha, x) : \alpha \in \kappa\}$ be a family of open sets containing x such that $\bigcap_\kappa cl(V(\alpha, x)) = \{x\}$. Let $L : \mathcal{P}(X) \rightarrow X$ be a choice function. Using transfinite induction, we will construct a sequence $\{H_\alpha : 0 \leq \alpha < \kappa^+\}$ of subsets of X such that for $0 \leq \alpha < \kappa^+$:

- (a) $H_0 = \{L(\emptyset)\}$;
- (b) if H_β is defined for $\beta < \alpha$, define H_α as follows:

$$f\left(\left(\bigcup_{\beta < \alpha} H_\beta\right) \cup \left\{L\left(X \setminus \bigcup_{x \in A} cl(V(x, g(x)))\right) : A \in \left[\bigcup_{\beta < \alpha} H_\beta\right]^{<\omega}, g : A \rightarrow \kappa\right\}\right).$$

Note that $|H_\alpha| \leq 2^\kappa$ for $0 \leq \alpha < \kappa^+$. Let $H = \bigcup \{H_\alpha : \alpha < \kappa^+\}$. It follows that $|H| \leq 2^\kappa$ and $f(H) \subseteq H$. Thus, $H = f(H)$ and if $\mathcal{F} \in [[H]^\kappa]^\kappa$, $a_{\mathcal{F}}^H(\mathcal{F}) \neq \emptyset$. By Lemmas 3.8 and 3.10, H is an H-set.

To show that $H = X$, assume that $q \notin H$. Since $\psi_c(X) \leq \kappa$, for each $x \in H$, there is $\alpha_x < \kappa$ such that $q \notin clV(\alpha_x, x)$. Using that H is H-set, there a finite subset $A \in [H]^{<\omega}$ such that $H \subseteq \bigcup_A cl(V(\alpha_x, x)) \subseteq X \setminus \{q\}$. Now choose $\alpha < \kappa^+$ such that $A \in [\bigcup_{\beta < \alpha} H_\beta]^{<\omega}$. By (b), $L(X \setminus \bigcup_{x \in A} cl(V(\alpha_x, x))) \in H_\alpha$ and it follows that $H_\alpha \setminus \bigcup_{x \in A} cl(V(\alpha_x, x)) \neq \emptyset$. This is a contradiction as $H_\alpha \subseteq H \subseteq \bigcup_{x \in A} cl(V(\alpha_x, x))$. \square

It follows from Theorem 3.11 that the cardinality of an H-closed space is at most $2^{\chi(X)}$. The Dow-Porter result given in Corollary 4.6 now follows, using a proof similar to the proof of that corollary. We see then two very different proofs of this result, one using open ultrafilters (which generalizes to a result for all Hausdorff spaces, Theorem 4.4) and the other using κ -nets [10] which can be reframed in terms of κ -filters as in Theorem 3.11. We note that in [15], Porter used a different type of open ultrafilter approach.

We present several examples.

Example 3.12. This example demonstrates that the converse of Lemma 3.8 is false, i.e., an ω -H-closed space X with a filter base $\mathcal{F} \in [[X]^{\leq \omega}]^{\leq \omega}$ such $a_c(\mathcal{F}) = \emptyset$. The space X is Tychonoff; so, $a_c(\mathcal{F}) = a_\theta(\mathcal{F}) = a(\mathcal{F})$.

Consider the partition $\{A_n : n \in \omega\}$ of ω where each A_n is infinite. Pick one point, say $a_n \in cl_{\beta\omega} A_n \setminus A_n$. Let $B = cl_{\beta\omega} \{a_n : n \in \omega\} \setminus \{a_n : n \in \omega\}$. We will show that the subspace $X = \beta\omega \setminus B$ is ω -H-closed but has a filter base $\mathcal{F} \in [[X]^{\leq \omega}]^{\leq \omega}$ such $a_c(\mathcal{F}) = \emptyset$. For $n \in \omega$, let $F_n = \{a_m : m \geq n\}$; thus, $\mathcal{F} = \{F_n : n \in \omega\} \in [[X]^{\leq \omega}]^{\leq \omega}$.

Let $x \in X$. Then, in $\beta\omega$, $B \cup \{a_n : n \in \omega\} \setminus \{x\}$ is compact and there are disjoint open sets U, V in $\beta\omega$ such that $B \cup \{a_n : n \in \omega\} \setminus \{x\} \subseteq U$ and $x \in V$. $U \setminus B$ and $V \setminus B$ are disjoint open sets in X such that $\{a_n : n \in \omega\} \setminus \{x\} \subseteq U \setminus B$ and $x \in V \setminus B$. If $x = a_m$ for some $m \in \omega$, then $F_{m+1} \cap cl_X(V \setminus B) = \emptyset$ and $x \notin a_c(\mathcal{F})$. If $x \notin \{a_n : n \in \omega\}$, then $F_0 \cap cl_X(V \setminus B) = \emptyset$ and $x \notin a_c(\mathcal{F})$. So in both cases, $x \notin a_c(\mathcal{F})$ and $a_c(\mathcal{F}) = \emptyset$.

To show that X is ω -H-closed (= feebly compact), it suffices to show that the Tychonoff space X is pseudocompact by 1.10(d)(2) in [17]. It suffices, by 1Q(6) in [17] to show that every infinite subset of ω is not closed in X . Let $C = \{b_n : n \in \omega\}$ be infinite subset of ω . As $cl_{\beta\omega} A_n \cap B = \emptyset$ for each $n \in \omega$ and $cl_{\beta\omega} (A_n \cap C) \setminus \omega \neq \emptyset$ whenever $A_n \cap C$ is infinite, it follows that $A_n \cap C$ is finite for each $n \in \omega$. Thus, by 4B(6) in [7], $\{a_n : n \in \omega\}$ and C are contained in disjoint cozero-sets (in an extremely disconnected space) and hence $B \cap cl_{\beta\omega} C = \emptyset$. It follows that $cl_{\beta\omega} C \setminus C \subseteq X$. \square

Example 3.13. The ψ space X is an example of a first countable, Tychonoff, pseudocompact space with a filter base $\mathcal{F} \in [[X]^{\leq \mathfrak{c}}]^{\leq \mathfrak{c}}$ such that $a_c(\mathcal{F}) = \emptyset$. Let $X = \omega \cup \mathcal{M}$ where \mathcal{M} is a maximal family of almost disjoint infinite subsets of ω and $U \subseteq X$ is open if $A \in \mathcal{M} \cap U$ implies there is a $F \in \omega^{<\omega}$ such that $A \setminus F \subseteq U$. It is well-known that X is first countable, locally compact, Tychonoff, pseudocompact space that is not countably compact and $|\mathcal{M}| = \mathfrak{c}$.

For $B \in [\mathcal{M}]^{<\omega}$, let $F_B = \mathcal{M} \setminus B$ and $\mathcal{F} = \{F_B : B \in [\mathcal{M}]^{<\omega}\}$. We will show that such $a_c(\mathcal{F}) = \emptyset$. Let $x \in X$. If $x \in \omega$, then $\{x\}$ is a clopen set disjoint from $\mathcal{M} \in \mathcal{F}$. If $x = A \in \mathcal{M}$, then if $\mathcal{M} \setminus \{A\} \in \mathcal{F}$, then as $cl_X(\{A\} \cup A) = \{A\} \cup A$, $cl_X(\{A\} \cup A) \cap (\mathcal{M} \setminus \{A\}) = \emptyset$. Thus, $a_c(\mathcal{F}) = \emptyset$. \square

For the space \mathbb{U} constructed in Example 3.6, the subspace $\{(n, 0) : n \in \mathbb{N}\} \cup \{\infty\}$ is an ω -H-set but not ω -H-closed.

4. A NEW CARDINALITY BOUND FOR HAUSDORFF SPACES.

Proposition 4.1. *If X is Hausdorff and $\psi_c(X) \leq \kappa$, then for all $x \in X$ there exists a family \mathcal{V} of open sets such that $|\mathcal{V}| \leq \kappa$ and*

$$\{x\} = \bigcap \mathcal{V} = \bigcap_{V \in \mathcal{V}} clV = \bigcap_{V \in \mathcal{V}} c(\widehat{V}).$$

Proof. Fix $x \in X$. As $\psi_c(X) \leq \kappa$, there exists a family \mathcal{V} of open sets such that $\{x\} = \bigcap \mathcal{V} = \bigcap_{V \in \mathcal{V}} clV$ and $|\mathcal{V}| \leq \kappa$. Suppose $y \neq x$. There exists $V \in \mathcal{V}$ such that $y \in X \setminus clV$. Let $W = X \setminus clV$ and suppose $y \in c(\widehat{V})$. Then,

$$\emptyset \neq \widehat{W} \cap \widehat{V} = \widehat{W \cap V} = \widehat{\emptyset} = \emptyset,$$

a contradiction. Thus $y \notin c(\widehat{V})$ and $\{x\} = \bigcap_{V \in \mathcal{V}} c(\widehat{V})$. As $\bigcap_{V \in \mathcal{V}} clV \subseteq \bigcap_{V \in \mathcal{V}} c(\widehat{V})$ by Proposition 2.7(c), it follows that

$$\{x\} = \bigcap \mathcal{V} = \bigcap_{V \in \mathcal{V}} clV = \bigcap_{V \in \mathcal{V}} c(\widehat{V}).$$

\square

Proposition 4.2. *If X is Hausdorff and $A \subseteq X$, then $|c(A)| \leq |A|^{t_c(X)\psi_c(X)}$.*

Proof. Let $\kappa = t_c(X)\psi_c(X)$. For each $x \in c(A)$, by Proposition 4.1 there exists a family \mathcal{V}_x of open sets such that $|\mathcal{V}_x| \leq \kappa$ and

$$\{x\} = \bigcap \mathcal{V}_x = \bigcap_{V \in \mathcal{V}_x} clV = \bigcap_{V \in \mathcal{V}_x} c(\widehat{V}).$$

As $t_c(X) \leq \kappa$, for all $x \in c(A)$ there exists $A(x) \in [A]^{\leq \kappa}$ such that $x \in c(A(x))$.

Define $\phi : c(A) \rightarrow [[A]^{\leq \kappa}]^{\leq \kappa}$ by

$$\phi(x) = \{\widehat{V} \cap A(x) : V \in \mathcal{V}_x\}.$$

Observe that $\phi(x) \in [[A]^{\leq \kappa}]^{\leq \kappa}$. Fix $x \in c(A)$. We will show that $x \in c(\widehat{V} \cap A(x))$ for all $V \in \mathcal{V}_x$. Let $V \in \mathcal{V}_x$ and let U be any open set containing x . As $x \in c(A(x))$, there exists $a \in A(x)$ such that $U \cap V \in \mathcal{U}_a$. Thus, $a \in \widehat{U \cap V} = \widehat{U} \cap \widehat{V}$ and it follows that $\widehat{U} \cap \widehat{V} \cap a(X) \neq \emptyset$. This shows $x \in c(\widehat{V} \cap A(x))$. Thus,

$$\{x\} \subseteq \bigcap_{V \in \mathcal{V}_x} c(\widehat{V} \cap A(x)) \subseteq \bigcap_{V \in \mathcal{V}_x} c(\widehat{V}) = \{x\},$$

where the second containment above follows from Proposition 2.7(a). Then $\{x\} = \bigcap_{V \in \mathcal{V}_x} c(\widehat{V} \cap A(x))$. Thus if $x \neq y$ then $\phi(x) \neq \phi(y)$, and ϕ is one-to-one. Therefore, $|c(A)| \leq |A|^\kappa$. \square

We turn now to our main result, a new bound for the cardinality of a Hausdorff space X . To establish this bound, we use the set-theoretic Theorem 3.1 from [10]. This theorem generalizes many closing-off arguments needed to prove cardinality bounds on topological spaces. For reference, we re-state the particular case of this theorem that is used here.

Theorem 4.3 (Hodel). *Let X be a set, κ be an infinite cardinal, $d : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ an operator on X , and for each $x \in X$ let $\{V(\alpha, x) : \alpha < \kappa\}$ be a collection of subsets of X . Assume the following:*

- (T) (tightness condition) *if $x \in d(H)$ then there exists $A \subseteq H$ with $|A| \leq \kappa$ such that $x \in d(A)$;*
- (C) (cardinality condition) *if $A \subseteq X$ with $|A| \leq \kappa$, then $|d(A)| \leq 2^\kappa$;*
- (C-S) (cover-separation condition) *if $H \neq \emptyset$, $d(H) \subseteq H$, and $q \notin H$, then there exists $A \subseteq H$ with $|A| \leq \kappa$ and a function $f : A \rightarrow \kappa$ such that $H \subseteq \bigcup_{x \in A} V(f(x), x)$ and $q \notin \bigcup_{x \in A} V(f(x), x)$.*

Then $|X| \leq 2^\kappa$.

Typically, the operator d used in Theorem 4.3 is either the standard closure operator cl , or in some instances the θ -closure cl_θ . We use the operator c .

Theorem 4.4. *If X is Hausdorff then $|X| \leq 2^{aL'(X)t_c(X)\psi_c(X)}$.*

Proof. Let $\kappa = aL'(X)t_c(X)\psi_c(X)$. As $\psi_c(X) \leq \kappa$, for all $x \in X$ there exists a family $\mathcal{W}_x = \{W(\alpha, x) : \alpha < \kappa\}$ of open sets such that $\{x\} = \bigcap \mathcal{W}_x = \bigcap_{W \in \mathcal{W}_x} cl W$. For all $x \in X$ and $\alpha < \kappa$, set $V(\alpha, x) = cl(W(\alpha, x))$. We verify the three conditions in Theorem 4.3, where the operator d is c . The (T) condition follows immediately as $t_c(X) \leq \kappa$, and (C) follows from Proposition 4.2. To verify (C-S), suppose $H \neq \emptyset$ satisfies $c(H) \subseteq H$. Then, $c(H) = H$, as $H \subseteq c(H)$ by Proposition 2.7(a), and H is c -closed. Let $q \notin H$. For all $a \in H$, there exist $\alpha_a < \kappa$ such that $q \notin cl(W(\alpha_a, a)) = V(\alpha_a, a)$. Define $f : A \rightarrow \kappa$ by $f(a) = \alpha_a$. Then $\{W(f(a), a) : a \in H\}$ is a cover of H by sets open in X . As $aL'(X) \leq \kappa$ and H is c -closed, there exists $A \in [H]^{\leq \kappa}$ such that $H \subseteq \bigcup_{a \in A} V(f(a), a)$. Since $q \notin \bigcup_{a \in A} V(f(a), a)$, we see that (C-S) is satisfied. By Theorem 4.3, $|X| \leq 2^\kappa$. \square

As $aL'(X) \leq aL_c(X)$ by Proposition 2.10 and $t_c(X)\psi_c(X) \leq \chi(X)$ by Proposition 2.12, we obtain the following Corollary 4.5. This is a slight weakening of the Bella-Cammaroto bound $2^{aL_c(X)t(X)\psi_c(X)}$ for Hausdorff spaces. While $aL'(X) \leq aL_c(X)$, it is unclear whether $t(X)$ and $t_c(X)$ are comparable for a non-regular space X , making it unclear whether $2^{aL'(X)t_c(X)\psi_c(X)}$ and $2^{aL_c(X)t(X)\psi_c(X)}$ are comparable.

Corollary 4.5. [Bella/Cammaroto] *If X is Hausdorff then $|X| \leq 2^{aL_c(X)\chi(X)}$.*

Corollary 4.6. [Dow/Porter] *If X is H-closed then $|X| \leq 2^{\psi_c(X)}$.*

Proof. The semiregularization $X(s)$ of X is also H-closed, and so by Corollary 3.3 and Corollary 3.5 it follows that $aL'(X_s) = \aleph_0$. Thus, by Theorem 4.4, we have that

$$|X| = |X_s| \leq 2^{t_c(X(s))\psi_c(X(s))} \leq 2^{\chi(X(s))} = 2^{\psi_c(X(s))} \leq 2^{\psi_c(X)},$$

where the second equality follows as $X(s)$ is minimal Hausdorff. \square

We see then that Theorem 4.4 leads to a common proof of the cardinality bound $2^{\chi(X)}$ for both H-closed spaces and Lindelöf spaces simultaneously. We can isolate the precise property \mathcal{P} that both H-closed spaces and Lindelöf spaces X share from which it follows from Theorem 4.4 that $|X| \leq 2^{\chi(X)}$. Property \mathcal{P} is the property that every open cover \mathcal{V} of X has a countable subfamily \mathcal{W} such that $X = \bigcup_{W \in \mathcal{W}} \widehat{W}$. That is, $L'(X) = \aleph_0$. In fact, the weaker property $aL'(X) = \aleph_0$ also suffices.

5. GENERALIZED H-CLOSED SPACES

The standard method of generalizing the concept of H-closed is to use the well-known cardinality invariant of almost Lindelöf – when $aL(X) \leq \aleph_0$, the space X is a generalized H-closed space. One of the main goals in this paper is seek generalized H-closed spaces for which it is possible to obtain a cardinality bound of X . A space X satisfying $a'L(X) \leq \aleph_0$ is another generalized H-closed space for which it is possible to obtain a cardinality bound of X (see Theorem 4.4). We used the concept of κ -H-closed, another generalized H-closed space, to obtain a cardinality bound of H-closed spaces (see Theorem 3.11). In this section, we examine three new generalized H-closed concepts with the common goal of obtaining a cardinality bound of a space.

Approach I.

The roots of our first generalized H-closed space can be traced back to the famous 1929 memoir (the Russian version of [1]) and uses a recent characterization of H-closed spaces by Osipov [14]. Alexandroff and Urysohn proved this property of H-closed spaces: If X is an H-closed space and $A \subseteq X$ is an infinite subset, there is a point $p \in X$ such that $|A| = |A \cap cl(U)|$ whenever $p \in U \in \tau(X)$.

Definition 5.1. Let X be a space and $A \subseteq X$. A point $p \in X$ is a Θ -complete accumulation point of A (we write $p \in \Theta CAP(A)$) if whenever $p \in U \in \tau(X)$, $|cl(U) \cap A| = |A|$. In particular, $cl_\theta A = (\Theta CAP(A)) \cup (cl_\theta A \setminus \Theta CAP(A))$.

An exciting new characterization of H-closed spaces using the concept of Θ -complete accumulation points was established in 2013 by Osipov [14].

Theorem 5.2. [Osipov] *Let X be a Hausdorff space and $A \subseteq X$ an infinite subset,*

- (a) *X is H-closed iff for each open cover \mathcal{C} of $\Theta CAP(A)$, there is a finite subfamily $\mathcal{F} \subseteq \mathcal{C}$ such that $|A \setminus int(cl(\cup \mathcal{F}))| < |A|$.*

(b) If X is H -closed space, then $\Theta CAP(A)$ is an H -set.

We start the process of generalizing H -closed spaces by expanding the notation ΘCAP .

Definition 5.3. Let X be an H -closed space, κ an infinite cardinal, and A an infinite subset. Let $cl_\theta^\kappa A$ denote $\{p \in X : |cl(U) \cap A| \geq \kappa \text{ for } p \in U \in \tau(X)\}$. Note that $cl_\theta^{|A|} A = \Theta CAP(A)$. If σ is an infinite cardinal and $\kappa \geq \sigma$, then $cl_\theta^\kappa(A) \subseteq cl_\theta^\sigma(A)$ and $cl_\theta^\omega(A)$ is the set of accumulation points of A .

Using a techniques similar to the proof of Theorem 5.2, it is possible to obtain this result at the κ level.:

Proposition 5.4. Let X be an H -closed space, κ an infinite cardinal, A an infinite subset, and \mathcal{C} an open cover of $cl_\theta^\kappa A$. There is a finite subfamily $\mathcal{D} \subseteq \mathcal{C}$ such that $|A \setminus int(cl(\cup \mathcal{D}))| < \kappa$.

Proof. Let \mathcal{C} be an open cover of $cl_\theta^\kappa A$. For $p \notin cl_\theta^\kappa A$, there is an open set U_p such that $p \in U_p$ and $|cl(U_p) \cap A| < \kappa$. Let $\mathcal{O} = \{U_p : p \notin cl_\theta^\kappa A\}$. As X is H -closed, there are finite subfamilies $\mathcal{D} \subseteq \mathcal{C}$ and $\mathcal{U} \subseteq \mathcal{O}$ such that $X = cl(\cup \mathcal{D}) \cup cl(\cup \mathcal{U})$. So, $cl(X \setminus cl(\cup \mathcal{D})) \subseteq cl(\cup \mathcal{U})$ implying that $X \setminus int(cl(\cup \mathcal{D})) \subseteq cl(\cup \mathcal{U})$. Since $|(cl(\cup \mathcal{U})) \cap A| < \kappa$, it follows that $|A \setminus int(cl(\cup \mathcal{D}))| < \kappa$. \square

Definition 5.5. Let X be a space and κ be infinite cardinal. A filter base \mathcal{F} on X is said to be κ -wide if $|cl_\theta(A)| \geq \kappa$ for each $A \in \mathcal{F}$. A space X is κwH -closed if for each κ -wide filter base \mathcal{F} on X , $a_\theta \mathcal{F} \neq \emptyset$.

Proposition 5.6. Let X be a space and κ be an infinite cardinal.

- (a) The space X is H -closed iff X is $\aleph_0 wH$ -closed.
- (b) Let X be a space and κ be infinite cardinal. X is κwH -closed iff for each κ -wide open filter base \mathcal{F} , $a_\theta \mathcal{F} \neq \emptyset$.

Proof. The proof of (a) is immediate. The proof of (b) follows the known result that if \mathcal{F} is a filter base on a space X , then the open filter base $\mathcal{G} = \{U \in \tau(X) : F \subseteq U \text{ for some } F \in \mathcal{F}\}$ has the property $a_\theta \mathcal{G} = a_\theta \mathcal{F}$. \square

Theorem 5.7. Let X be space and κ an infinite cardinal. The space X is κwH -closed iff for every subset $A \subseteq X$ where $\kappa \leq |A|$ and \mathcal{C} is an open cover of $cl_\theta^\kappa A$, there is a finite subfamily $\mathcal{B} \subseteq \mathcal{C}$ such that $|A \setminus int(cl(\cup \mathcal{B}))| < \kappa$.

Proof. Suppose X is κwH -closed and $A \subseteq X$ where $\kappa \leq |A|$ and \mathcal{C} is an open cover of $cl_\theta^\kappa A$. For each $p \notin cl_\theta^\kappa A$, there is $p \in U_p \in \tau(X)$ such that $|cl(U_p) \cap A| < \kappa$. Let $\mathcal{E} = \{U_p : p \notin cl_\theta^\kappa A\}$. Assume, by way of contradiction, that for each finite subfamily \mathcal{B} of \mathcal{C} , $|A \setminus int(cl(\cup \mathcal{B}))| \geq \kappa$.

Claim: $\mathcal{F} = \{X \setminus cl(\cup \mathcal{A}) : \mathcal{A} \in [\mathcal{E} \cup \mathcal{C}]^{<\omega}\}$ is an κ -wide filter base such that $|cl(V)| \geq \kappa$ for each $V \in \mathcal{F}$.

Proof of Claim. Let $\mathcal{A} \in [\mathcal{C} \cup \mathcal{E}]^{<\omega}$. Then there are finite subfamilies $\mathcal{B} \subseteq \mathcal{C}$ and $\mathcal{D} \subseteq \mathcal{E}$ such that $\mathcal{A} = (\cup \mathcal{B}) \cup (\cup \mathcal{D})$. It suffices to show that $|cl(X \setminus cl(\cup \mathcal{A}))| =$

$|cl(X \setminus cl((\cup \mathcal{B}) \cup (\cup \mathcal{D})))| \geq \kappa$. Note that $|cl(\cup \mathcal{D}) \cap A| < \kappa$ implies $|(X \setminus cl(\cup \mathcal{D})) \cap A| = |A \setminus cl(\cup \mathcal{D})| = |A| \geq \kappa$. Also, note that $|cl(X \setminus cl(\cup \mathcal{B}))| \geq \kappa$ by the assumption. We have:

$$\begin{aligned} cl(X \setminus cl((\cup \mathcal{B}) \cup (\cup \mathcal{D}))) &= cl((X \setminus cl(\cup \mathcal{B})) \cap (X \setminus cl(\cup \mathcal{D}))) \\ &= cl(cl(X \setminus cl(\cup \mathcal{B})) \cap (X \setminus cl(\cup \mathcal{D}))) \\ &= cl((X \setminus int(cl(\cup \mathcal{B}))) \cap (X \setminus cl(\cup \mathcal{D}))) \\ &\supseteq cl((X \setminus int(cl(\cup \mathcal{B}))) \cap (A \setminus cl(\cup \mathcal{D}))) \\ &= cl((A \setminus int(cl(\cup \mathcal{B}))) \setminus cl(\cup \mathcal{D}))). \end{aligned}$$

This shows that $|cl(X \setminus cl(\cup A))| \geq |cl((A \setminus int(cl(\cup \mathcal{B}))) \setminus cl(\cup \mathcal{D}))| \geq \kappa$ and \mathcal{F} is a κ -wide filter base. \square

As X is κ WH-closed, there is some $p \in a\mathcal{F}$. If $p \in cl_\theta^\kappa A$, there is $U \in \mathcal{C}$ such that $p \in U$. Thus, $X \setminus cl(U) \in \mathcal{F}$ and $p \notin cl(X \setminus cl(U))$; so, $p \notin a(\mathcal{F})$. On the other hand, if $p \notin a\mathcal{F}$, there is $U \in \mathcal{E}$ such that $p \in U$. Again, $X \setminus cl(U) \in \mathcal{F}$, $p \notin cl(X \setminus cl(U))$, and $p \notin a(\mathcal{F})$. Hence, $a(\mathcal{F}) = \emptyset$. This contradicts the hypothesis.

To show the converse, let \mathcal{F} be a free κ -wide open filter base on X . Let $U \in \mathcal{F}$ such $|cl(U)|$ is minimum. We will apply the condition in the statement of the theorem to the set $cl(U)$. In particular, $|cl(U)| \geq \kappa$. If $p \in cl_\theta^\kappa(cl(U))$, there is $V_p \in \mathcal{F}$ such that $p \notin cl(V_p)$ and $V_p \subseteq U$. Then $p \in X \setminus cl(V_p)$ and $|cl(X \setminus cl(V_p)) \cap cl(U)| \geq \kappa$. Let $\mathcal{C} = \{X \setminus cl(V_p) : p \in cl_\theta^\kappa(cl(U))\}$. By the hypothesis of the converse, there is a finite subfamily $\mathcal{B} \subseteq \mathcal{C}$ such that $|cl(U) \setminus int(cl(\cup \mathcal{B}))| < \kappa$. For $V = \cap \{V_p : X \setminus cl(V_p) \in \mathcal{B}\} \cap U$, note that $V \in \mathcal{F}$, $cl(V) \subseteq cl(U)$ and $V \cap (\cup \mathcal{B}) = \emptyset$. It follows that $clV \cap int(cl(\cup \mathcal{B})) = \emptyset$. This shows that $clV \subseteq clU \setminus int(cl(\cup \mathcal{B}))$. That is, $V \in \mathcal{F}$ and $|clV| < \kappa$, a contradiction. \square

As corollaries of Theorems 5.2 and 5.7, we have the following results.

Corollary 5.8. *Let X be a space and κ be an infinite cardinal.*

- (a) *The space X is H -closed iff X is κ WH-closed for all infinite $\kappa \leq |X|$.*
- (b) *If X is a κ WH-closed space, $A \subseteq X$ such that $|A| \geq \kappa$, and \mathcal{F} is a κ -wide open filter base on X that meets $cl_\theta^\kappa A$, then $a(\mathcal{F}) \cap cl_\theta^\kappa A \neq \emptyset$.*
- (c) *If X is κ WH-closed and $A \subseteq X$ such that $|A| \geq \kappa$, then $cl_\theta^\kappa A \neq \emptyset$.*

Proof. The proof of (a) is straightforward. To prove (b), let \mathcal{F} be a κ -wide open filter base on X that meets $cl_\theta^\kappa A$. Assume that $a(\mathcal{F}) \cap cl_\theta^\kappa A = \emptyset$. Let $U \in \mathcal{F}$ such $|cl(U) \cap A|$ is minimum. Note that $|cl(U) \cap A| \geq \kappa$. If $p \in cl_\theta^\kappa(A)$, there is $V_p \in \mathcal{F}$ such that $p \notin cl(V_p)$ and $V_p \subseteq U$. Then $p \in X \setminus cl(V_p)$ and $|cl(X \setminus cl(V_p)) \cap A| \geq \kappa$. Let $\mathcal{C} = \{X \setminus cl(V_p) : p \in cl_\theta^\kappa A\}$. As X is κ WH-closed, there is a finite subfamily $\mathcal{B} \subseteq \mathcal{C}$ such that $|A \setminus int(cl(\cup \mathcal{B}))| < \kappa$. Now $V = \cap \{V_p : X \setminus cl(V_p) \in \mathcal{B}\} \cap U \in \mathcal{F}$. Also, $clV \subseteq clU$ and $V \cap (\cup \mathcal{B}) = \emptyset$. It follows that $clV \cap int(cl(\cup \mathcal{B})) = \emptyset$. This shows that $clV \subseteq clU \setminus int(cl(\cup \mathcal{B}))$. Thus, $clV \cap A \subseteq (clU \setminus int(cl(\cup \mathcal{B}))) \cap A = (clU \cap A) \setminus int(cl(\cup \mathcal{B})) \subseteq A \setminus int(cl(\cup \mathcal{B}))$. This implies there is a $V \in \mathcal{F}$ such that $|clV \cap A| < \kappa$, a contradiction.

To show (c), assume that $cl_\theta^\kappa A = \emptyset$. For each $p \in X$, there is an open set $p \in U_p \in \tau(X)$ such that $|clU_p \cap A| < \kappa$. Then $\mathcal{U} = \{U_p : p \in X\}$ is an open

cover of X (and $cl_\theta^\kappa A$). There is a finite $\mathcal{V} \subseteq \mathcal{U}$ such that $|A \setminus \text{int}(cl(\cup \mathcal{V}))| < \kappa$. Let $B = A \setminus \text{int}(cl(\cup \mathcal{V}))$. Then $A \subseteq B \cup \text{int}(cl(\cup \mathcal{V})) \subseteq B \cup cl(\cup \mathcal{V})$. It follows that $A \subseteq B \cup (\cup_{V \in \mathcal{V}} cl(V) \cap A)$. Thus, $|A| \leq |B| + \sum_{V \in \mathcal{V}} |cl(V) \cap A| < \kappa$. This is a contradiction. \square

The study of κ wH-closed spaces is a new approach to understanding the theory of H-closed spaces by using the width of a filter base. The width is a measure of the size of the closure of the elements of a filter base.

There is still the question of obtaining a cardinality bound of κ wH-closed spaces. We are able to obtain such a result only for κ wH-closed spaces with a dense subset of isolated points. We start by stating a well-known result that is similar to 4.2.

Lemma 5.9. *Let κ be an infinite cardinal and $\chi(X) \leq \kappa$. For $A \subseteq X$, $|cl(A)| \leq |A|^\kappa$.*

Theorem 5.10. *Let κ be an infinite cardinal and X a κ wH-closed space with a dense set of isolated points and $\chi(X) \leq \kappa$. Then $|X| \leq 2^\kappa$.*

Proof. For each $p \in X$, let $V(p) = \{V(\alpha, p) : \alpha < \kappa\}$ be an open neighborhood base at p , and for $B \subseteq X$ and $f : B \rightarrow \kappa$, let $V(f, B) = \bigcup_{p \in B} V(f(p), p)$. Let D be the set of isolated points of X . If $p \in D$, we let $V(\alpha, p) = \{p\}$ for all $\alpha \in \kappa$.

Let $H : \mathcal{P}(D) \rightarrow D$ be a choice function and $A_0 = H(\emptyset)$. We will inductively define A_α for $\alpha < \kappa^+$. For $\alpha < \kappa^+$, suppose A_β is defined for $\beta < \alpha$. Let

$$A_\beta = \bigcup_{\alpha < \beta} A_\alpha \cup \bigcup \{H(D \setminus V(g, B)) : B \in [cl(\bigcup_{\alpha < \beta} A_\alpha)]^{\leq \kappa}, g : B \rightarrow \kappa\}.$$

By induction, $|A_\alpha| \leq 2^\kappa$; let $C = \bigcup_{\alpha < \kappa^+} A_\alpha$. It follows that $|C| \leq 2^\kappa$. By Lemma 5.9, for $A = cl(C)$, $|A| \leq 2^\kappa$. As $C \subseteq D$ and is open, we also have that $A = cl_\theta(C)$. Also, note that as C is an increasing chain over κ^+ and $\chi(X) \leq \kappa$, $A = cl(C) = \bigcup_{\alpha < \kappa^+} cl(A_\alpha)$.

To apply Theorem 5.7, we need to show that $|C| \geq \kappa$. Suppose that $|C| \leq \kappa$. As $C = \bigcup_{\alpha < \kappa^+} A_\alpha$, there is $\alpha < \kappa$ such that $C \subseteq A_\alpha$. Let $g : C \rightarrow \kappa : p \mapsto 0$. Then $C = V(g, C)$ and it follows that $D \setminus V(C, g) = \emptyset$. Thus, $D \subseteq C$, $|D| \leq \kappa$, and it follows that $|X| \leq 2^\kappa$ and we are done. We are reduced the case when $|C| \geq \kappa$.

To finish the proof of the theorem, we will prove that $X = cl(C) = A$ by showing that $D \subseteq C$. Let $d \in D \setminus C = D \setminus A$. For each $p \in A$, there is some $U_p = V(\alpha_p, p) \in V(p)$ such that $d \notin cl(U_p)$. Then $\mathcal{C} = \{U_p : p \in A\}$ is an open cover of $A = cl_\theta C \supseteq cl_\theta^\kappa C$; note that $cl_\theta^\kappa C \subseteq X \setminus D$. By Theorem 5.7, there is a finite subfamily \mathcal{B} of \mathcal{C} such that $|C \setminus \text{int}(cl(\cup \mathcal{B}))| < \kappa$. There is a finite subset $F \subseteq A$ such that $\mathcal{B} = \{U_p : p \in F\}$. For $U = \cup_{p \in F} U_p \cap D$, $C \setminus \text{int}(cl(\cup \mathcal{B})) = C \setminus U$. Define $h : C \setminus U \cup F \rightarrow \kappa : p \mapsto \alpha_p$; it follows that $C \subseteq V(h, C \setminus U \cup F)$.

Now, as $|C \setminus U \cup F| < \kappa$ and $cl(C) = \bigcup_{\alpha < \kappa^+} cl(A_\alpha)$, there is some $\beta < \kappa^+$, such that $C \setminus U \cup F \subseteq cl(A_\beta)$. As $d \notin V(h, C \setminus U \cup F)$, $D \setminus V(h, C \setminus U \cup F) \neq \emptyset$. Thus, $H(D \setminus V(C \setminus U \cup F, h)) \in A_{\beta+1} \subseteq C$, a contradiction. This completes the proof that $D \subseteq C$ and finishes the proof. \square

We ask whether the above Theorem 5.10 is true without the hypothesis that X has a dense set of isolated points.

Question 5.11. *If κ is an infinite cardinal and X a κ wH-closed space such that $\chi(X) \leq \kappa$, is $|X| \leq 2^\kappa$?*

Approach II.

The application of κ -H-closed spaces in Theorem 3.11 to obtain a cardinality bound of H-closed spaces provides another approach to studying H-closed spaces by using “thin” filter bases where a filter base is a member of $[[X]^{\leq \kappa}]^{\leq \kappa}$. This technique is another way of measuring the width of a filter base and provides our second path in defining generalized H-closed.

Definition 5.12. Let X be a space, κ an infinite cardinal, and $A \subseteq X$ such that $|A| \geq \kappa$. Define $c^\kappa(A) = \{x \in X : \text{if } x \in U \in \tau(X), \text{ then } |\hat{U} \cap A| \geq \kappa\}$. A space X is κ H'-closed if $A \subseteq X$, $|A| \geq \kappa$, and \mathcal{U} is an open cover of $c^\kappa(A)$, there is a finite subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $|A \setminus \cup_{V \in \mathcal{V}} \hat{V}| < \kappa$. If $|X| < \kappa$, it follows that X is κ H'-closed.

Proposition 5.13. *A space X is \aleph_0 H'-closed iff X is H-closed.*

Proof. Suppose X is \aleph_0 H'-closed. Let \mathcal{U} be open cover of X . We can assume that $|X| \geq \aleph_0$. Then \mathcal{U} covers $c^{\aleph_0}(X)$. Then there is a finite $\mathcal{V} \subseteq \mathcal{U}$ such that $|X \setminus \cup_{V \in \mathcal{V}} \hat{V}| < \aleph_0$. By Theorem 3.4, X is H-closed. Conversely, suppose X is H-closed. Let $A \subseteq X$ such that $|A| \geq \aleph_0$. Let \mathcal{U} be open cover of $c^{\aleph_0}(A)$. For each $p \notin c^{\aleph_0}(A)$, there is $p \in U_p \in \tau(X)$ such that $|\hat{U}_p \cap A| < \aleph_0$. Now, $\{U_p : p \notin c^{\aleph_0}(A)\} \cup \mathcal{U}$ is open cover of X . There is a finite $B \subseteq X \setminus c^{\aleph_0}(A)$ and a finite $\mathcal{V} \subseteq \mathcal{U}$ such that $X = \cup_{p \in B} \hat{U}_p \cup \cup_{V \in \mathcal{V}} \hat{V}$. Thus, $X \setminus \cup_{V \in \mathcal{V}} \hat{V} \subseteq \cup_{p \in B} \hat{U}_p$ and $|A \setminus \cup_{V \in \mathcal{V}} \hat{V}| \leq |\cup_{p \in B} (\hat{U}_p \cap A)| \leq \sum_{p \in B} |\hat{U}_p \cap A| < \aleph_0$. This shows that X is \aleph_0 H'-closed. \square

Proposition 5.14. *Let κ be infinite cardinal and X be κ H'-closed. Then $aL'(X) < \kappa$.*

Proof. Let A be c -closed. If $|A| < \kappa$, then $aL'(A, X) < \kappa$. So, suppose that $|A| \geq \kappa$. Let \mathcal{U} be open cover of A . For each $p \notin A$, there is $p \in U_p \in \tau(X)$ such that $\hat{U}_p \cap A = \emptyset$. Now, $\{U_p : p \notin A\} \cup \mathcal{U}$ is open cover of X . There is a finite $B \subseteq X \setminus A$ and a finite $\mathcal{V} \subseteq \mathcal{U}$ such that $|X \setminus (\cup_{p \in B} \hat{U}_p \cup \cup_{V \in \mathcal{V}} \hat{V})| < \kappa$. Now, $|A \setminus (\cup_{p \in B} \hat{U}_p \cup \cup_{V \in \mathcal{V}} \hat{V})| = |A \setminus (\cup_{V \in \mathcal{V}} \hat{V})| < \kappa$. Thus, there is $\mathcal{W} \subseteq \mathcal{U}$ such that $|\mathcal{W}| < \kappa$ and $A \setminus (\cup_{V \in \mathcal{V}} \hat{V}) \subseteq \cup_{W \in \mathcal{W}} \hat{W} \subseteq \cup_{W \in \mathcal{W}} \hat{W}$. Therefore, $A \subseteq \cup_{V \in \mathcal{V}} \hat{V} \cup \cup_{W \in \mathcal{W}} \hat{W}$ and $|\mathcal{V} \cup \mathcal{W}| < \kappa$. So, $aL'(A, X) < \kappa$. \square

Definition 5.15. For a space X , define $z(X) = \inf\{\kappa \geq \aleph_0 : X \text{ is } \kappa\text{H'-closed}\}$.

By Proposition 5.14 and Theorem 4.4, we have the following two results.

Corollary 5.16. *For a space X ,*

- (a) $aL'(X)^+ \leq z(X)$ and

$$(b) |X| \leq 2^{z(X)t_c(X)\psi_c(X)}.$$

Remark. This last corollary is an indication that the concept of $\kappa H'$ -closed is subsummed by the theory using c -closure and aL' .

Approach III.

In this third approach to generalized H-closed spaces, we modify the concept defined in Definition 5.12 in Path II.

Definition 5.17. Let X be a space, κ an infinite cardinal, and $A \subseteq X$ such that $|A| \geq \kappa$. Define $c^\kappa(A) = \{x \in X : \text{if } x \in U \in \tau(X), \text{ then } |\hat{U} \cap A| \geq \kappa\}$. A space X is $\kappa H''$ -closed if $A \subseteq X$, $|A| \geq \kappa$, and \mathcal{U} is an open cover of $c^\kappa(A)$, there is a subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $|\mathcal{V}| \leq \kappa$ and $|A \setminus \bigcup_{V \in \mathcal{V}} \hat{V}| < \kappa$. In particular, if $|X| < \kappa$, then X is $\kappa H''$ -closed.

Using essentially the same proof as the proof of Proposition 5.14, we obtain the following result.

Proposition 5.18. *Let κ be infinite cardinal and X be $\kappa H''$ -closed. Then $aL'(X) \leq \kappa$.*

Definition 5.19. For a space X , define $z''(X) = \inf\{\kappa \geq \aleph_0 : X \text{ is } \kappa H''\text{-closed}\}$.

By Proposition 5.18 and Theorem 4.4, we have the following two results.

Corollary 5.20. *For a space X ,*

- (a) $aL'(X) \leq z''(X)$ and
- (b) $|X| \leq 2^{z''(X)t_c(X)\psi_c(X)}.$

Remark. Using Approach III, 5.20(a) is sharper than 5.16(a). However, the price is that the counterpart to Proposition 5.13 is not true; that is, in the case when $\kappa = \aleph_0$, we do not necessarily get H-closed. In fact, any countable space is $\aleph_0 H''$ -closed.

REFERENCES

- [1] P.S. Alexandroff and P.S. Urysohn, *Mémoire sur les espaces topologiques compacts*, Nederl. Akad. Wetensch. Proc. Sr. A (1929), 1-96.
- [2] A. V. Arhangel'ski, *On the cardinality of bicomacta satisfying the first axiom of countability*, Soviet Math. Dokl. **10** (1969) 951-955.
- [3] A. Bella, F. Cammaroto, *On the cardinality of Urysohn spaces*, Canad. Math. Bull. **31** (1988) 153-158.
- [4] A. Bella, I. V. Yaschenko, *Embeddings into first countable spaces with H-closed like properties*, Topology Appl. **83** (1998), no. 1, 5361.
- [5] A. Dow, J.R. Porter, *Cardinalities of H-closed spaces*, Topology Proc. **7** (1982), no. 1, 2750.
- [6] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, second ed., 1989.
- [7] L. Gillman, M. Jerison, *Rings of Continuous Functions*, University Series in Higher Mathematics, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960.
- [8] A. Gryzlov, *Two theorems on the cardinality of topological spaces*, Soviet Math. Dokl. **21** (1980) 506509.
- [9] R.E. Hodel, *Cardinal functions I*, in: K. Kunen, J.E. Vaughan (Eds.), *Handbook of Set-Theoretic Topology*, North-Holland, Amsterdam, 1984, pp. 161.

- [10] R.E. Hodel, *Arhangel'skii's solution to Alexandroff's problem: A survey*, Topology Appl. **153** (2006) 2199–2217.
- [11] J.E. Joseph, *Multifunctions and Cluster Sets*, Proc. Amer. Math. Soc. **74** (1979) 329–337.
- [12] I. Juhász, *Cardinal Functions in Topology - Ten Years Later*, Math. Centre Tracts **123**, Amsterdam, 1980.
- [13] G. A. Kirtadze, Different types of completeness of topological spaces, Mat Sb. **50**(1960) 67-90.
- [14] A. V. Osipov, Some Properties of Minimal $S(\alpha)$ and $S(\alpha)FC$, Top Proc **42** (2013) 91-105.
- [15] J. R. Porter, *Extensions of Discrete Spaces*, Papers on general topology and applications, Ann. New York Acad. Sci., **704** (1993), 290–295.
- [16] J. R. Porter and J. Vermeer, *Spaces with Coarser Minimal Hausdorff Topologies*, Trans. Amer. Math. Soc., **289** (1985), no. 1, 59–71.
- [17] J. R. Porter and R.G. Woods, *Extensions and absolutes of Hausdorff spaces*, Springer-Verlag, New York, 1988.
- [18] J. R. Porter and R.G. Woods, *Minimally extremally disconnected Hausdorff spaces*, General Topology Appl., **8** (1978), no. 1, 9–26.
- [19] B. Šapirovič, *Canonical sets and character. Density and weight in compact spaces*, Soviet Math. Dokl. **15** (1974) 1282-1287.

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